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FORCED CONVECTION IN A CHANNEL WITH WALL HEAT CAPACITY AND WITH WALL HEATING VARIABLE WITH AXIAL POSITION AND TIME

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Abstract—Heat transfer to a fluid with constant properties is analysed for forced convection between two parallel plates. The plates have a finite heat capacity, and heat is supplied to them in an arbitrary manner with both time and axial position. The wall is assumed sufficiently thin or highly conducting so that the temperature variation through the wall thickness can be neglected. The fluid temperature is variable over the channel cross section, but the fluid velocity is assumed constant throughout the channel (slug-flow condition). The energy equation is laminar. A general method of solution is given and then some illustrative examples are carried out. These include uniform wall heating that varies sinusoidally in time, and heating varying sinusoidally with axial distance and exponentially in time.

NOMENCLATURE

- A, C, D, constants;
- a, half width of spacing between parallel plates;
- b, thickness of channel wall;
- c_p, specific heat at constant pressure;
- F_n, eigenvalue, $n\pi$;
- k, thermal conductivity of fluid;
- L, dimensionless length of channel, $4(l/a)/RePr$;
- l, length of channel;
- N, ratio of wall and fluid heat capacities, $b\rho_w c_{p,w}/a\rho c_p$;
- Pr, Prandtl number of fluid, ν/α;
- Q, dimensionless heat flux, q/q_r;
- q, local heat addition per unit area at channel walls; q_r, reference value;
- Re, Reynolds number, $\bar{u}4a/\nu$;
- T, dimensionless temperature, $(t - t_0)k/q_r a$;
- t, temperature; t_0, temperature of fluid entering channel, (a constant);
- u, fluid velocity; $\bar{u}$ mean fluid velocity over channel cross section;
- X, dimensionless co-ordinate, $4(x/a)/RePr$;
- x, axial distance from entrance of heated section of channel.

Greek symbols

- α, thermal diffusivity of fluid, $k/\rho c_p$;
- γ_n, eigenvalue, see equation (10);
- Θ, dimensionless time, $\tau\nu/a^2Pr = \tau\alpha/a^2$;
- θ, dummy Θ variable;
- ν, kinematic viscosity of fluid;
- ρ, fluid density;
- τ, time.

Subscripts

- g, refers to heat generated in (or supplied to) wall;
- m, refers to the mth column of the characteristic line mesh;
- r, refers to rth row of the characteristic line mesh (except in q_r where r denotes reference value);
- w, refers to wall.

INTRODUCTION

IN THE flow channels of nuclear reactors or nuclear rocket engines, transient heating conditions are encountered during power changes, startup, and shut down. In addition to having time variations, the wall heat flux can also change with axial position along the channel because of spatial variations in fuel loading or neutron flux. This type of situation has promoted interest in the general class of problems involving channel flows with arbitrary transient heating at the walls. When analysing this type of problem some simplifying assumptions are generally required. In one type of approach the temperature and velocity distributions in the fluid and the temperature distribution in the wall are all assumed one-dimensional. A constant convective heat-transfer coefficient is then specified to relate the local wall and fluid temperatures. Some examples of this type of analysis can be found in [1, 2, 3, 4, 5].

In other instances the wall heat capacity has been neglected. This has provided sufficient simplification to make it possible to solve for the temperature distribution in the fluid which eliminates the restriction of specifying a heat-transfer coefficient. For simple wall boundary conditions the correct velocity distribution can also be accounted for, e.g. [6]. However, for more complex conditions it has been necessary to utilize the slug flow assumption, that is consider the velocity one dimensional, e.g. [7].

In [7] the solution was provided for slug flow in a channel with zero wall heat capacity and with wall heating that can vary arbitrarily with both position along the channel and time. No simplifying assumptions were made with respect to the fluid temperature distribution. With this formulation as a beginning the present paper will show how solutions can be found which include wall heat capacity and have arbitrary wall heating with position and time. During a transient, part of the energy supplied to the channel walls is stored within the walls while the remainder is transferred into the fluid. Hence the heat flux that the fluid receives from the wall is generally a function of both distance and time; this is the case treated in [7]. However, with wall heat capacity, the heat flow to the fluid is an unknown function as it depends on the transient heat storage within the wall. As shown in the analysis, by coupling the wall and fluid heat-transfer equations it is possible to determine the amount of heat flow into the fluid and from this the wall temperatures are found. After the general method is discussed, some illustrative examples are carried out. These include a wall heat flux that oscillates sinusoidally with time, and one that varies sinusoidally with axial position and exponentially with time. The latter is typical of a reactor runaway transient.

ANALYSIS

The geometry under consideration is illustrated in Fig. 1. Fluid with a uniform temperature t_0 enters a channel consisting of two parallel plates. The fluid velocity is assumed to

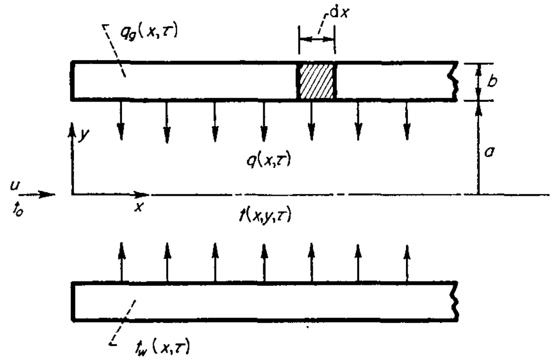

FIG. 1. Parallel plate channel with wall thickness, b, and arbitrary wall heat generation, q_g.

remain constant throughout the channel (slug flow assumption), that is, the variation of velocity with the y co-ordinate is not taken into account and the flow is incompressible. At or within the channel walls there is a specified heat generation $q_g(x, \tau)$ which can vary arbitrarily with both axial position and time. Part of this heat generation will be stored in the walls by virtue of the wall heat capacity, and the remainder is the quantity $q(x, \tau)$ that is transferred from the walls to the fluid. The walls are assumed to be sufficiently thin so that at any time, the wall temperature at each axial location can be assumed constant over the thickness b. The temperature in the fluid, however, is a function of location y within the cross section as well as x and τ. For all of the cases treated here the walls and fluid are considered to be initially isothermal at the entering fluid temperature t_0. Then at time equal to zero the wall heat generation is suddenly applied equally to both walls. Since the wall heat generation is specified, the quantity of physical interest to be computed in the analysis is the wall temperature as a function of position and time. The conditions of an initially isothermal channel with zero heating are not as restrictive as they may appear. For example, consider a transient beginning from an initial steady state condition with uniform wall heating. This can be conveniently reduced to the conditions of the present computational procedure by considering a fictitious earlier time when the uniform heating is applied to an isothermal channel. The resulting transient is computed until steady state is achieved, and the variation in wall heat generation is then continued to study the desired wall heat flux transient.

Basic equations

The first step in the analysis is to form a heat balance on an element of the wall as shown by the shaded volume in Fig. 1. The heat balance states that the change of energy stored within the element is equal to the heat generated in the wall minus the heat transferred to the fluid. This gives

$$b\rho_w c_{p,\,w}\frac{\partial t_w}{\partial \tau} = q_g(x, \tau) - q(x, \tau). \tag{1}$$

By using the dimensionless variables defined in the Nomenclature, equation (1) can be placed in the form

$$\frac{\partial T_w}{\partial \Theta} = \frac{1}{N} Q_g(X, \Theta) - \frac{1}{N} Q(X, \Theta). \tag{1a}$$

The heat quantities Q_g and Q have been nondimensionalized relative to q_r which is any convenient constant reference wall heat generation. The parameter N is equal to $b\rho_w c_{p,\,w}/a\rho c_p$ which is the ratio of the wall heat capacity to the heat capacity of the fluid. Hence, $N \to 0$ provides the limiting case of negligible wall capacity. Equation (1a) can be integrated to obtain T_w, and the boundary condition is employed that the channel is initially isothermal, $T_w(X, 0) = 0$. This gives,

$$T_w(X, \Theta) = \frac{1}{N}\int_0^{\Theta} Q_g(X, \theta)\,\mathrm{d}\theta - \frac{1}{N}\int_0^{\Theta} Q(X, \theta)\,\mathrm{d}\theta. \tag{2}$$

The next step is to consider how the conditions at the inside surface of the walls effect the heat transfer within the fluid. The fluid is being subjected to a wall heating Q which is a function of both axial position and time so a solution is needed for a channel flow with this type of arbitrary wall heating. For slug flow in a channel this solution is given in [7] and can be used here without modification. Since the derivation is given in detail in [7], it does not seem worthwhile to repeat it here, and the reader is referred to [7] for further information. The solution is in two parts. For the region where $X \geqslant \Theta$ the wall temperature distribution is given by:

$$T_w(X, \Theta) = \int_0^{\Theta} Q(X - \Theta + \theta, \theta)\,\mathrm{d}\theta$$
$$+ 2\sum_{n=1}^{\infty} e^{-F_n^2\Theta}\int_0^{\Theta} Q(X - \Theta + \theta, \theta)\, e^{F_n^2\theta}\,\mathrm{d}\theta. \tag{3a}$$

For the region where $X \leqslant \Theta$ the following expression applies:

$$T_w(X, \Theta) = \int_{\Theta - X}^{\Theta} Q(X - \Theta + \theta, \theta)\,\mathrm{d}\theta$$
$$+ 2\sum_{n=1}^{\infty} e^{-F_n^2\Theta}\int_{\Theta - X}^{\Theta} Q(X - \Theta + \theta, \theta)\, e^{F_n^2\theta}\,\mathrm{d}\theta. \tag{3b}$$

Equations (2) and (3) are to be solved simultaneously for $T_w(X, \Theta)$ and the most convenient procedure is to first eliminate T_w by substituting (2) into (3). This gives the following relations for Q:

For $X \geqslant \Theta$,

$$\left.\begin{aligned} &\frac{1}{N}\int_0^{\Theta} Q_g(X, \theta)\,\mathrm{d}\theta - \frac{1}{N}\int_0^{\Theta} Q(X, \theta)\,\mathrm{d}\theta \\ &= \int_0^{\Theta} Q(X - \Theta + \theta, \theta)\,\mathrm{d}\theta \\ &+ 2\sum_{n=1}^{\infty} e^{-F_n^2\Theta}\int_0^{\Theta} Q(X - \Theta + \theta, \theta)\, e^{F_n^2\theta}\,\mathrm{d}\theta. \end{aligned}\right\} \tag{4a}$$

For $X \leqslant \Theta$,

$$\left.\begin{aligned} &\frac{1}{N}\int_0^{\Theta} Q_g(X, \theta)\,\mathrm{d}\theta - \frac{1}{N}\int_0^{\Theta} Q(X, \theta)\,\mathrm{d}\theta \\ &= \int_{\Theta - X}^{\Theta} Q(X - \Theta + \theta, \theta)\,\mathrm{d}\theta \\ &+ 2\sum_{n=1}^{\infty} e^{-F_n^2\Theta}\int_{\Theta - X}^{\Theta} Q(X - \Theta + \theta, \theta)\, e^{F_n^2\theta}\,\mathrm{d}\theta. \end{aligned}\right\} \tag{4b}$$

Since $Q_g(X, \Theta)$ is a specified quantity, equation (4) can be solved for the single unknown $Q(X, \Theta)$. After this is found, the wall temperature, which is the quantity of physical interest, is found from the integrations in (2). The next step, then, is to provide a means for solving (4).

In Fig. 2a the X, Θ plane is shown. A 45° diagonal line divides the area into the two regions in each of which one of equations (4) apply depending on whether X is greater or less than Θ. The significance of the diagonal line can be shown by considering a fluid element that enters the tube at time $\tau = 0$. This element reaches location x at $\tau = x/\bar{u}$, and the dimensionless variables have been chosen such that this is equivalent to $\Theta = X$ (that is, the velocity in the dimensionless system is unity). Hence, the line $X = \Theta$ in Fig. 2a traces the path of the fluid that starts from the channel entrance at the initiation of the transient. The regions below and above the diagonal are physically different. For $X > \Theta$ the channel contains fluid that was already within the channel when the transient began, while for $X < \Theta$ the fluid entered the channel after the transient had started. Since the solution method is the same for both of equations (4), only (4a) will be discussed for the moment. An integral such as $\int_0^\Theta Q(X, \theta)\, d\theta$ is carried out at constant X with θ as a dummy time variable. Hence, this type of integral is along a vertical line in Fig. 2a. An integral of the type $\int_0^\Theta Q(X - \Theta + \theta, \theta)\, d\theta$ is carried out along a 45° diagonal line and integrates the heat flow following the fluid element that arrives at location X at time Θ. The diagonal and vertical lines form a grid of characteristic lines in the $X - \Theta$ plane. The integration along vertical lines accounts for the heat transferred from the wall at each location, while the integrals along the diagonals provide the convective energy moving with the fluid. It will be shown that the desired values of Q can be computed at successive time intervals in the $X - \Theta$ plane by knowing the previous time history. Before this is done the boundary conditions must be discussed.

Boundary conditions

At $\Theta = 0$ the wall heat generation is initiated. At this instant the wall temperature is still equal to the fluid temperature so there is no heat transfer from the wall to the fluid. Hence, at $\Theta = 0$: $Q = 0$ for all $X > 0$. At $X = 0$ the fluid is always maintained at a constant entering temperature and the heat-transfer coefficient is infinite as the thermal boundary layer has zero

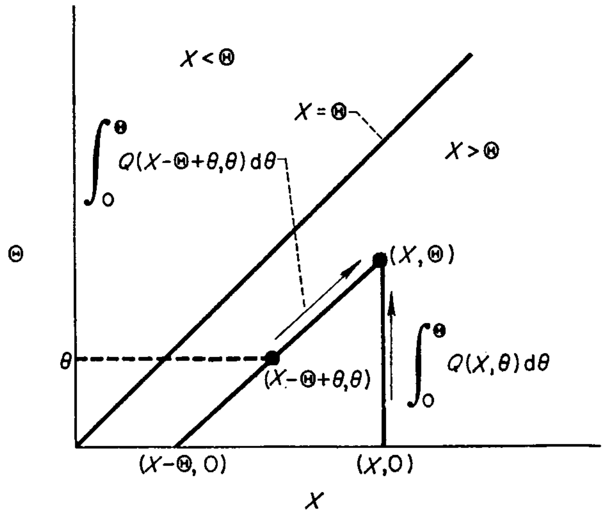

FIG. 2a. Heat flow integrals along characteristic lines in the $X - \Theta$ plane.

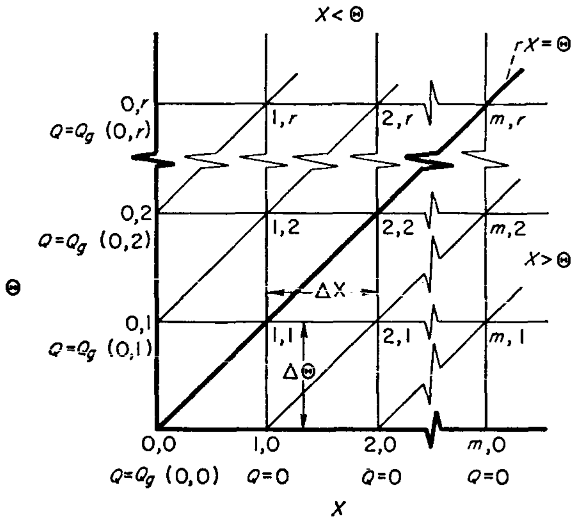

FIG. 2b. Grid points and boundary conditions for numerical solution.

thickness. Consequently, at $X = 0$ the wall temperature always remains fixed at the entering fluid temperature and $\partial T_w/\partial\Theta|_{X=0} = 0$. From (1a) this gives the condition, at $X = 0$: $Q = Q_g$ for all Θ greater than zero. The single point $X = \Theta = 0$ is singular in that either condition could apply, but this does not lead to any difficulty, as either condition can be arbitrarily selected. This is similar to computing the temperatures near the corner of a rectangular plate when the two edges are at different temperatures.

Numerical solution

The solution to equations (4) can be best illustrated by considering a few specific points on the grid shown in Fig. 2b. It will be necessary to set the integrals in (4) into finite difference form, and for all except the second integral on the right side the trapezoidal rule is used. This approximation would not be very accurate for the second integral on the right side due to the rapid variation of the exponential function $e^{F_n^2\theta}$ for large F_n. For this integral a good approximation is found by noting that the function Q would be expected to be fairly smooth and hence segments of the Q variation can be approximated quite well by straight lines. In the range between θ_p and θ_{p+1} the function $Q(\theta)$ is approximated by

$$Q(\theta) = Q_p + (\theta - \theta_p)\left(\frac{Q_{p+1} - Q_p}{\theta_{p+1} - \theta_p}\right). \tag{5}$$

Then the exponential variation in the integrand can be integrated out analytically. The integral

$$\int_{\theta_p}^{\theta_{p+1}} Q(\theta)\, e^{F_n^2\theta}\, d\theta$$

becomes

$$\int_{\theta_p}^{\theta_{p+1}} \left[Q_p + (\theta - \theta_p)\left(\frac{Q_{p+1} - Q_p}{\theta_{p+1} - \theta_p}\right)\right] e^{F_n^2\theta}\, d\theta,$$

and this integrates to

$$\frac{1}{F_n^2}\left[-Q_p\, e^{F_n^2\theta_p} + Q_{p+1}\, e^{F_n^2\theta_{p+1}} + \frac{1}{F_n^2}\left(\frac{Q_{p+1} - Q_p}{\Delta\theta}\right)\left(e^{F_n^2\theta_p} - e^{F_n^2\theta_{p+1}}\right)\right]. \tag{6}$$

Consider for example the point (2,1) in Fig. 2b. This is in the region where $X > \Theta$ so (4a) is used. By using the trapezoidal rule and equation (6) this is placed in the finite difference form:

$$\left.\begin{aligned}&\frac{\Delta\Theta}{2N}(Q_{g_{2,1}}+Q_{g_{2,0}})-\frac{\Delta\Theta}{2N}(Q_{2,1}+Q_{2,0})\\&=\frac{\Delta\Theta}{2}(Q_{2,1}+Q_{1,0})\\&+2\sum_{n=1}^{\infty}\frac{e^{-F_n^2\Theta_{2,1}}}{F_n^2}\Big[-Q_{1,0}e^{F_n^2\Theta_{1,0}}\\&+Q_{2,1}e^{F_n^2\Theta_{2,1}}+\frac{1}{F_n^2}\left(\frac{Q_{2,1}-Q_{1,0}}{\Delta\Theta}\right)\\&\left(e^{F_n^2\Theta_{1,0}}-e^{F_n^2\Theta_{2,1}}\right)\Big].\end{aligned}\right\}\quad(7a)$$

The boundary condition along the X-axis is then applied which gives $Q_{1,0}=Q_{2,0}=0$, $\Theta_{1,0}=\Theta_{2,0}=0$, and it is noted that $\Theta_{2,1}=\Delta\Theta$. Applying these relations, (7a) can be placed in the form for $Q_{2,1}$:

$$Q_{2,1}=\frac{Q_{g_{2,1}}+Q_{g_{2,0}}}{1+N+\dfrac{4N}{\Delta\Theta}\displaystyle\sum_{n=1}^{\infty}\frac{1}{F_n^2}\left(1-\frac{1}{F_n^2\Delta\Theta}\right)+\frac{e^{-F_n^2\Delta\Theta}}{F_n^4\Delta\Theta}}.$$

This is further simplified by noting that

$$\sum_{n=1}^{\infty}\frac{1}{F_n^2}=\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{6}$$

and

$$\sum_{n=1}^{\infty}\frac{1}{F_n^4}=\frac{1}{90},$$

so that finally,

$$Q_{2,1}=\frac{Q_{g_{2,1}}+Q_{g_{2,0}}}{1+N+\dfrac{2N}{\Delta\Theta}\left(\dfrac{1}{3}-\dfrac{1}{45\Delta\Theta}+\dfrac{2}{\Delta\Theta}\displaystyle\sum_{n=1}^{\infty}\frac{e^{-F_n^2\Delta\Theta}}{F_n^4}\right)}.\quad(7b)$$

Equation (7b) shows that the points in the second horizontal row in Fig. 2b are found from those in the first row. In a similar fashion the points in the third row are found from those in the first two rows. Hence, the solution propagates forward row by row in time. One more example will be given; point (2,2). This is on the line $X=\Theta$ and either of equations (4) will apply. There is a singularity at point (0,0) in that Q can either equal zero or $Q_g(0,0)$. The latter alternative was arbitrarily selected; hence, points along $X=\Theta$ will belong to the class computed from (4b). To obtain $Q_{2,2}$, (4b) is placed in finite difference form with the aid of (6),

$$\begin{aligned}&\frac{\Delta\Theta}{2N}(Q_{g_{2,0}}+2Q_{g_{2,1}}+Q_{g_{2,2}})\\&-\frac{\Delta\Theta}{2N}(Q_{2,0}+2Q_{2,1}+Q_{2,2})\\&=\frac{\Delta\Theta}{2}(Q_{0,0}+2Q_{1,1}+Q_{2,2})\\&+2\sum_{n=1}^{\infty}\frac{e^{-F_n^2\Theta_{2,2}}}{F_n^2}\Big[-Q_{0,0}{}^{F_n^2\Theta_{0,0}}+Q_{1,1}e^{F_n^2\Theta_{1,1}}\\&+\frac{1}{F_n^2}\left(\frac{Q_{1,1}-Q_{0,0}}{\Delta\Theta}\right)\times(e^{F_n^2\Theta_{0,0}}-e^{F_n^2\Theta_{1,1}})\\&-Q_{1,1}e^{F_n^2\Theta_{1,1}}+Q_{2,2}e^{F_n^2\Theta_{2,2}}\\&+\frac{1}{F_n^2}\left(\frac{Q_{2,2}-Q_{1,1}}{\Delta\Theta}\right)(e^{F_n^2\Theta_{1,1}}-e^{F_n^2\Theta_{2,2}})\Big].\end{aligned}\quad(8a)$$

The boundary conditions are applied that $Q_{2,0}=0$, $\Theta_{0,0}=0$, $Q_{0,0}=Q_g(0,0)$, $\Theta_{1,1}=\Delta\Theta$, and $\Theta_{2,2}=2\Delta\Theta$. Equation (8a) can then be arranged in the form

$$\begin{aligned}&Q_{2,2}\Big[\frac{\Delta\Theta}{2}\left(\frac{1}{N}+1\right)+\frac{1}{3}-\frac{1}{45\Delta\Theta}\\&+\frac{2}{\Delta\Theta}\sum_{n=1}^{\infty}\frac{e^{-F_n^2\Delta\Theta}}{F_n^4}\Big]=\frac{\Delta\Theta}{2N}\\&(Q_{g_{2,0}}+2Q_{g_{2,1}}+Q_{g_{2,2}})\\&-\frac{\Delta\Theta}{N}Q_{2,1}-\frac{\Delta\Theta}{2}(Q_{g_{0,0}}+2Q_{1,1})\\&-2\sum_{n=1}^{\infty}\frac{1}{F_n^4\Delta\Theta}\{Q_{g_{0,0}}[e^{-F_n^2 2\Delta\Theta}(-F_n^2\Delta\Theta\\&-1)+e^{-F_n^2\Delta\Theta}]+Q_{1,1}(e^{-F_n^2 2\Delta\Theta}\\&-2e^{-F_n^2\Delta\Theta}+1)\}.\end{aligned}\quad(8b)$$

In a similar fashion, the finite difference equations can be placed in a general form for a point in the mth column and rth row. For $m > r$, the points are in the region where $X > \Theta$, while for $m \leqslant r$ the region under study is $X \leqslant \Theta$. The general relations, which are listed in the Appendix, are used to compute the Q's row by row throughout the X, Θ region under consideration. After the Q values are found they are integrated according to equation (2) to find T_w. The quantity Q_g is specified for each X, Θ point on the grid and hence can have any desired variation with axial position and time. Now that the method of analysis has been described specific examples will be given and discussed.

UNIFORM WALL HEAT ADDITION ($Q_g = 1$)

As a first example, a channel is considered which is initially isothermal and then has a uniform heat addition suddenly applied at its walls. The uniform heat addition is selected as the reference heat flux q_r, so Q_g is equal to unity. The results are shown in Fig. 3 for various N values. When $N = 0$ the wall has zero heat capacity, while for $N = 1$ the heat capacity of the wall is equal to that of the fluid. Parts (a) and (b) of the figure show the variation of wall temperature with time at fixed locations along the channel. The general shape of the curves is indicated by the set of arrows which show how the curve is followed for $N = 0{\cdot}2$ and $X = 0{\cdot}6$. After heating is applied, the wall temperature rises for a time and then levels out at a steady state value. The behavior of the curves can be explained quite well in physical terms as follows.

Consider first the limiting case of zero wall capacity, $N = 0$. This has been previously studied in [8] and the solution given analytically as:

For $\Theta \leqslant X$,

$$T_w = \Theta + \frac{1}{3} - 2\sum_{n=1}^{\infty}\frac{e^{-F_n^2\Theta}}{F_n^2}. \qquad (9a)$$

For $\Theta \geqslant X$,

$$T_w = X + \frac{1}{3} - 2\sum_{n=1}^{\infty}\frac{e^{-F_n^2 X}}{F_n^2}. \qquad (9b)$$

As discussed previously, if a position X is considered along the channel, for the time interval $\Theta < X$ the fluid starting at the entrance of the channel at $\Theta = 0$ has not yet reached X. The nondimensional variables are defined in such a way that the velocity in the dimensionless system is unity, so that this fluid reaches X when $\Theta = X$. As a result the region where $X > \Theta$ behaves like a channel which is infinitely long in both directions since until $\Theta = X$ the location at X has not yet been signaled that the channel has an entrance. When the wall heating is uniform, the heat convection in the region $X > \Theta$ is identically zero since the heat carried into a differential length of the channel is equal to that carried out by virtue of the behavior like a doubly infinite channel. Hence for $\Theta \leqslant X$ the heat transfer is only by conduction and (9a) is the same as that for the sudden application of heat to the surfaces of a solid slab of thickness equal to the channel width $2a$. After a time $\Theta = X$ has elapsed, the fluid traveling from the channel entrance reaches X and the wall temperature is stabilized at the steady state value.

When $N > 0$, there is still an initial period of pure condition at each X location until Θ becomes equal to X. This conduction solution corresponds to that from a slab (the channel wall) with uniform heat generation and uniform spatial temperature distribution, to an adjacent slab (the fluid) of another material. The conduction solution for this case is obtained from [9]:

$$T_w = \frac{1}{N+1}\left[\Theta + \frac{1}{2} - \frac{3N+1}{6(N+1)}\right] - 2\sum_{n=1}^{\infty}\frac{e^{-\gamma_n^2\Theta}}{\gamma_n^2(\gamma_n^2 N^2 + N + 1)} \qquad (10)$$

where the eigenvalues γ_n are found from $\gamma_n \cot \gamma_n = -1/N$. Equation (10) reduces to (9a) when $N \to 0$. For a given N the curves for all X follow the results of equation (10) until $\Theta = X$. Then the curves branch off and adjust toward the steady state values. The steady state temperatures are the same as those reached for $N = 0$. This arises from the fact that when there is no further change taking place in the

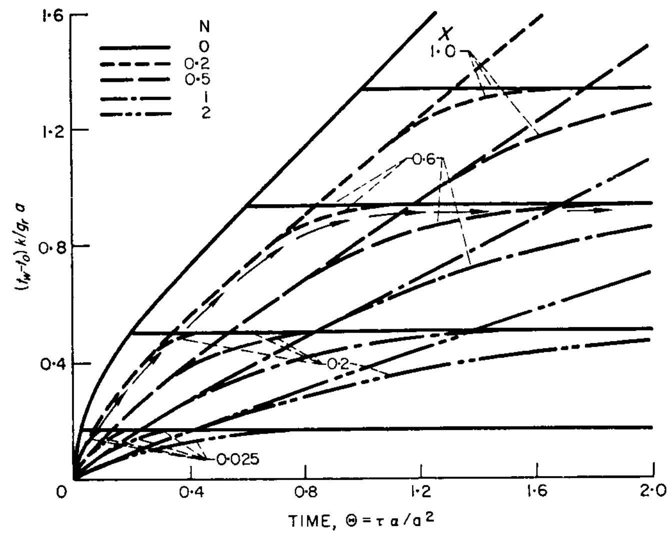

FIG. 3a. Wall temperature response for five wall heat capacities after a sudden application of uniform wall heat flux, $Q_g = 1$ (arrows explain path of curve for the example $N = 0{\cdot}2$, $X = 0{\cdot}6$).

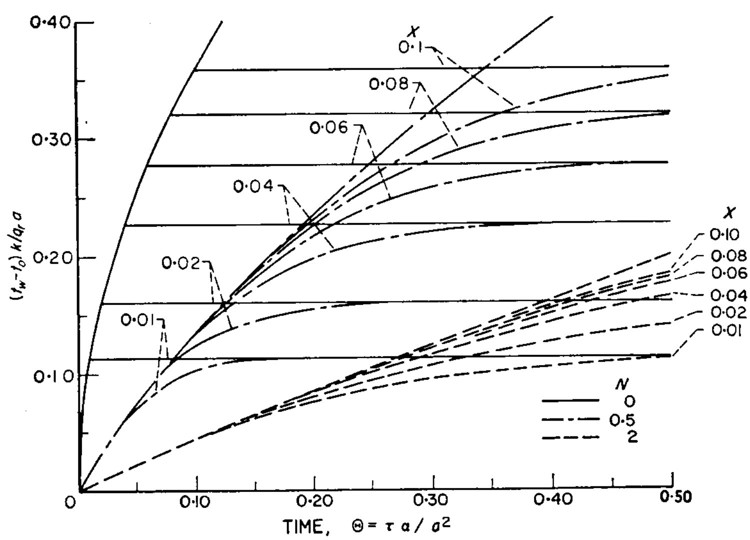

FIG. 3b. Wall temperature response for small X and Θ for three wall heat capacities after a sudden application of uniform wall heat flux, $Q_g = 1$.

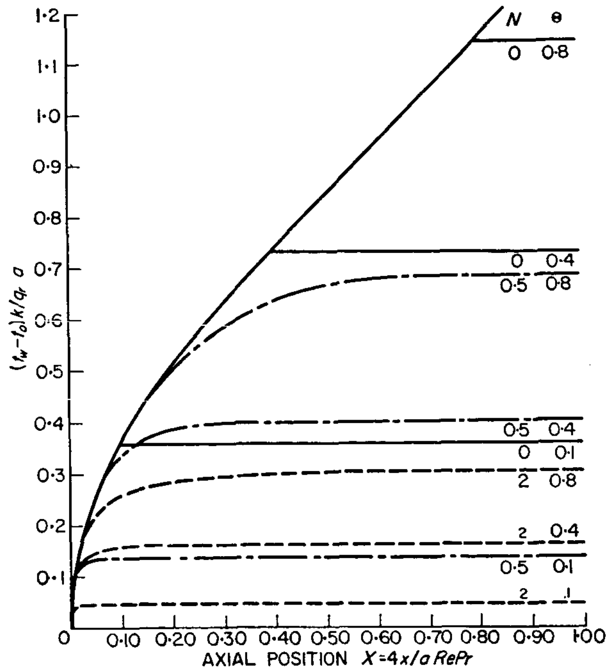

FIG. 3c. Wall temperature distributions at various times following the sudden application of a uniform wall heat flux, $Q_g = 1$.

wall temperature, all of the heat generated in the wall is being transferred to the fluid. This is the same condition on the fluid as for $N = 0$. The figure indicates how the wall temperatures for this transient can be obtained simply in an approximate way. The results for $N = 0$ can be computed without difficulty from (9). This gives the steady state values applicable for all N. For any $N \neq 0$ the initial transient curve can be found from (10). Then the complete solution for any X is obtained reasonably well by fairing the initial curve into the steady state value.

Fig. 3c is a cross plot giving the temperature distribution along the channel length at various times. At each time, for small X part of the channel has reached steady state and hence each curve initially follows along the steady state envelope line which is computed from (9b). At larger X, so that $X > \Theta$, the wall temperature is independent of X as there is only one-dimensional conduction taking place in this region.

HEAT FLUX A LINEAR FUNCTION OF X AND Θ ($Q_g = X\Theta$)

The next example is for a wall heating that is linear in both X and Θ. For zero wall heat capacity ($N = 0$) the solution can be found by substituting $Q_g = X\Theta$ for Q in (3). This results in:

For $X \geqslant \Theta$,

$$T_w = \frac{X\Theta^2}{2} - \frac{\Theta^3}{6}$$

$$+ 2\sum_{n=1}^{\infty} \frac{1}{F_n^4}\Bigg[(X - \Theta)(F_n^2\Theta - 1)$$

$$+ \frac{F_n^4\Theta^2 - 2F_n^2\Theta + 2}{F_n^2} - e^{-F_n^2\Theta}\left(\frac{2}{F_n^2} - X + \Theta\right)\Bigg]. \tag{11a}$$

For $X \leqslant \Theta$,

$$T_w = \frac{\Theta X^2}{2} - \frac{X^3}{6} + 2 \sum_{n=1}^{\infty} \frac{1}{F_n^4} \Big[(\Theta - X)(F_n^2 X - 1) + \frac{F_n^4 X^2 - 2F_n^2 X + 2}{F_n^2} - e^{-F_n^2 X}\Big(\frac{2}{F_n^2} - \Theta + X\Big)\Big]. \tag{11b}$$

These relations have been plotted as solid lines in Fig. 4 which gives the axial temperature distribution at several different times. As expected from the form of the heat input, the wall temperature increases with both position and time. When $N > 0$ the present analysis provides curves that are similar in shape to the zero capacity results except that the time response is reduced because of heat absorption within the wall.

HEAT FLUX UNIFORM WITH POSITION AND SINUSOIDAL WITH TIME ($Q_g = 1 + C \sin 2\pi A\Theta$)

In this example the heat flux is uniform along the length of the channel, but oscillates sinusoidally with time. The case of zero wall heat capacity ($N = 0$) is again considered first. To obtain the wall temperature transients for this case the expression for Q_g is substituted into (3) which yields:

For $X \geqslant \Theta$,

$$T_w = \Theta + \frac{C}{2\pi A}(1 - \cos 2\pi A\Theta) + \frac{1}{3} - 2 \sum_{n=1}^{\infty} \Big[\frac{e^{-F_n^2 \Theta}}{F_n^2} - \frac{C}{F_n^4 + (2\pi A)^2}(F_n^2 \sin 2\pi A\Theta - 2\pi A \cos 2\pi A\Theta + 2\pi A\, e^{-F_n^2 \Theta})\Big]. \tag{12a}$$

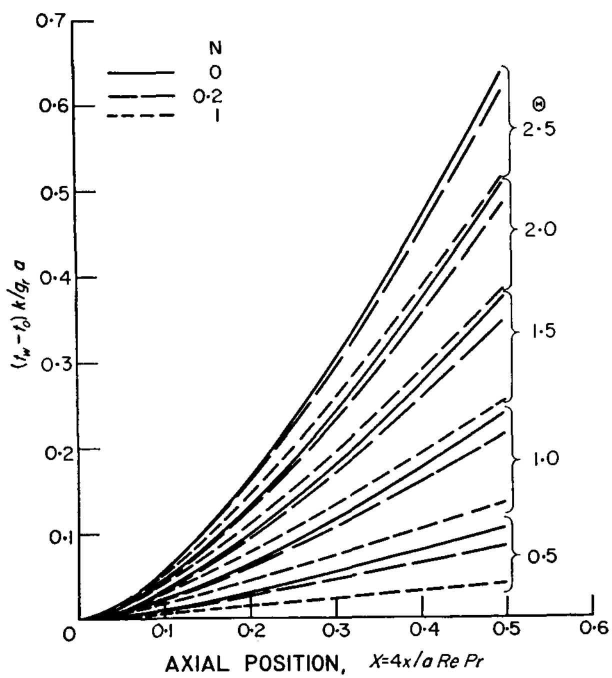

FIG. 4. Transient wall temperature response for the heat flux variation $Q_g = X\Theta$ and three different wall heat capacities.

For $X \leqslant \Theta$,

$$T_w = X + \frac{C}{2\pi A}\left[\cos 2\pi A(\Theta - X) - \cos 2\pi A\Theta\right]$$

$$+ \frac{1}{3} - 2\sum_{n=1}^{\infty}\left(\frac{e^{-F_n^2 X}}{F_n^2} - \frac{C}{F_n^4 + (2\pi A)^2}\right.$$

$$\{F_n^2 \sin 2\pi A\Theta - 2\pi A \cos 2\pi A\Theta + e^{-F_n^2 X}$$

$$\left.[F_n^2 \sin 2\pi A(\Theta - X) - 2\pi A \cos 2\pi A(\Theta - X)]\}\right). \tag{12b}$$

A numerical example has been plotted in Fig. 5 for $C = 1$ and $A = 2$. When $C = 1$ the amplitude of the wall heating oscillates between zero and 2. The frequency is such that a cycle is completed for every interval of $\Theta = 0{\cdot}5$. The figure shows the wall temperature behavior with time at a few different axial positions along the channel. There is an initial transient period of pure conduction during which all of the curves for $N = 0$ follow along the same line. Then when $\Theta = X$ each curve moves away from the common line and adjusts toward a steady oscillatory behavior. The wall temperatures are higher for larger X values because of the rise of the fluid temperature in the axial direction.

When $N > 0$ the time response of the wall temperature is diminished. For $N = 0{\cdot}5$ the behavior is similar to the $N = 0$ case except that the amplitude of the temperature fluctuations has been decreased. When the wall capacity is further increased to $N = 2$, the wall is no longer able to follow the heating fluctuations and the oscillations are completely damped out. The transient response then becomes the same as that in Fig. 3 where a steady heating was applied.

HEAT FLUX SINUSOIDAL WITH POSITION AND EXPONENTIAL WITH TIME

$$\left[Q_g = (e^{D\Theta} - 1)\sin\frac{\pi X}{L}\right]$$

This type of transient simulates a runaway power transient that can occur in a nuclear reactor. It is assumed for simplicity that the power has an elementary sinusoidal distribution in the axial direction. A numerical example is shown in Figs. 6a and 6b for a dimensionless length $L = 0{\cdot}5$ and a reciprocal period $D = 0{\cdot}2$. For $N = 0$ the wall temperatures are again found directly from (3) by substituting Q_g for Q. All of the curves shown in Figs. 6 are for $\Theta \geqslant X$ so

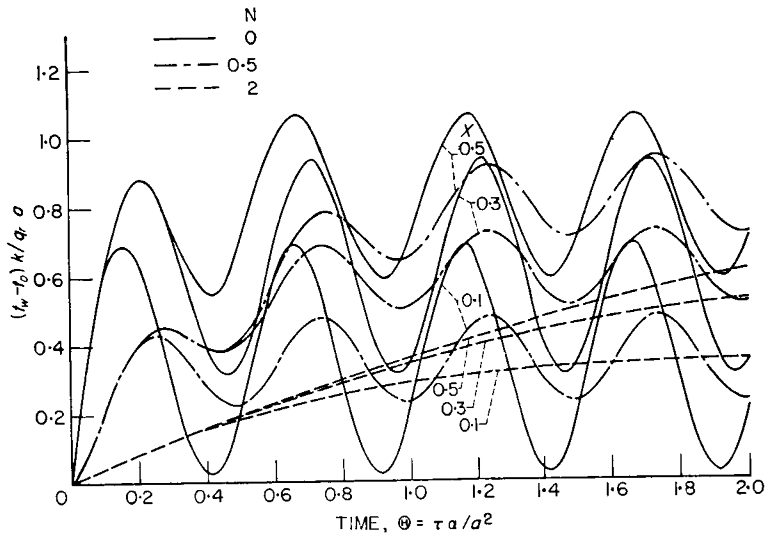

FIG. 5. Wall temperature response after the sudden application of the wall heat flux, $Q_g = 1 + \sin 4\pi\Theta$.

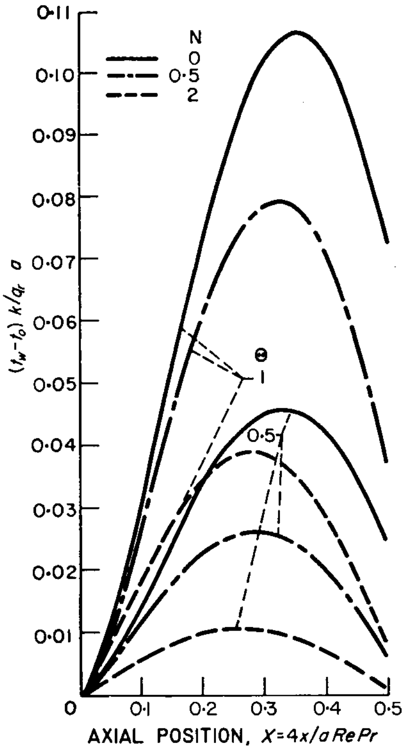

FIG. 6a. Wall temperature response for a transient simulating a nuclear reactor runaway. $Q_g = (e^{0\cdot 2\Theta} - 1)\sin \pi X/0\cdot 5$.

only (3b) is used. Carrying out the integration for $N = 0$

$$T_w = \frac{1}{D^2 + (\pi/L)^2}\left[e^{D\Theta}\left(D \sin\frac{\pi X}{L} - \frac{\pi}{L}\cos\frac{\pi X}{L}\right) + \frac{\pi}{L}e^{D(\Theta - X)}\right] + \frac{L}{\pi}\left(\cos\frac{\pi X}{L} - 1\right)$$

$$+ 2\sum_{n=1}^{\infty}\frac{1}{(D + F_n^2)^2 + (\pi/L)^2}\left\{e^{D\Theta}\left[(D + F_n^2)\sin\frac{\pi X}{L} - \frac{\pi}{L}\cos\frac{\pi X}{L}\right] + \frac{\pi}{L}e^{-F_n^2X + D(\Theta - X)}\right\}$$

$$- \frac{1}{F_n^4 + (\pi/L)^2}\left(F_n^2 \sin\frac{\pi X}{L} - \frac{\pi}{L}\cos\frac{\pi X}{L} + \frac{\pi}{L}e^{-F_n^2X}\right).$$

The results are shown in Figs. 6 for T_w as a function of X at five Θ values between 0·5 and 2·5. For each Θ, the wall temperatures reach a peak a little past the mid point of the channel, and then decrease near the end of the channel because of the diminished heat generation in that region.

With wall heat capacity included, solutions are shown for $N = 0{\cdot}5$ and 2. As the wall heat capacity is increased a greater amount of energy is stored in the walls and the time response of the channel is reduced. Consequently, for a given Θ the wall temperatures are much lower for larger N values.

For zero wall heat capacity, this type of transient has been compared in [7] with solutions obtained by assuming a constant heat-transfer coefficient. The simplified theory gave a fairly good qualitative indication of the system behavior for the parameters selected. The assumption of a constant heat-transfer

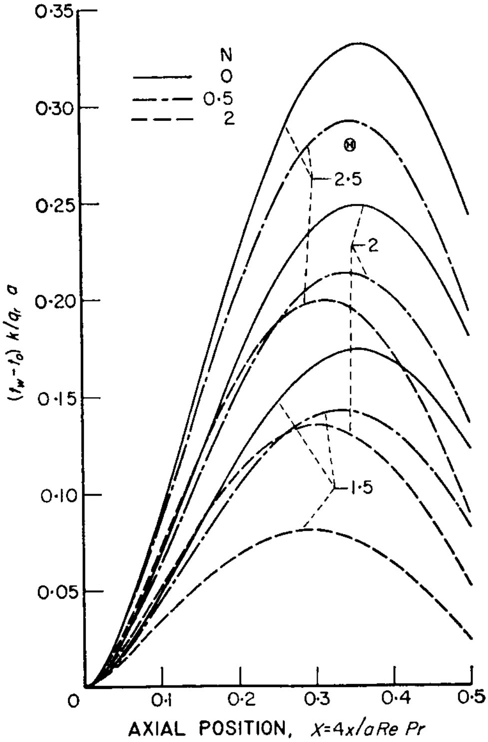

FIG. 6b. Wall temperature response for a transient simulating a nuclear reactor runaway. $Q_g = (e^{0\cdot 2\Theta} - 1)\sin \pi X/0\cdot 5$.

coefficient should be better as N is increased, as this reduces the magnitudes of the transient effects and the behavior becomes more quasi-steady.

CONCLUSIONS

A general method has been given for obtaining transient temperatures in a channel following the sudden application of a wall heat flux that can vary arbitrarily in the axial direction along the channel and with time. The method involves coupling the heat-transfer behavior within the fluid to that in the wall and solving the resultant equations by integrating along a grid of characteristics lines. The principal assumptions are a uniform fluid velocity throughout the channel and a constant wall temperature through the thickness of the wall at each axial position. The temperature profiles within the fluid are accounted for, which eliminates the need for assuming a convective heat-transfer coefficient. Thus effects such as thermal entrance regions and unsteadiness in the heat-transfer coefficient have been included. To demonstrate the method several illustrative examples are carried out for various types of transients and the results for finite wall heat capacity are compared with those for zero capacity. The capacity has a very substantial effect in reducing the time response of the surface temperatures.

The equations given here were derived on the basis that the entire channel is initially isothermal at the entering fluid temperature. This is not really a restrictive assumption since any initial condition can be conveniently treated without altering the computational method by starting the transient at a fictitious earlier time when the channel is isothermal. For example, if it is desired to begin from a steady state with uniform wall heating the whole process is started at an earlier time when a uniform flux is applied, and then after sufficient time for steady state to be reached the desired transient flux is imposed. The time scale is then shifted so zero at the instant the desired transient was applied.

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APPENDIX

Below are given the general relations for computing Q at any X, Θ location designated by point m,r in Fig. 2b.

For $m > r$,

$$Q_{m,r} = \frac{1}{K}\left\{\frac{\Delta\Theta}{2N}\left(Q_{g_{m,0}} + 2\sum_{s=1}^{r-1} Q_{g_{m,s}} + Q_{g_{m,r}}\right) - \frac{\Delta\Theta}{N}\sum_{t=1}^{r-1} Q_{m,t} - \Delta\Theta\,(Q_{m-r+1,1} + Q_{m-r+2,2} + \ldots + Q_{m-1,\,r-1}) - 2\sum_{n=1}^{\infty}\frac{1}{F_n^4\,\Delta\Theta}\,[Q_{m-r+1,1}\,(e^{-F_n^2 r\Delta\Theta} - 2e^{-F_n^2(r-1)\Delta\Theta} + e^{-F_n^2(r-2)\Delta\Theta}) + Q_{m-r+2,2}\,(e^{-F_n^2(r-1)\Delta\Theta} - 2e^{-F_n^2(r-2)\Delta\Theta} + e^{-F_n^2(r-3)\Delta\Theta}) + \ldots + Q_{m-1,\,r-1}\,(e^{-F_n^2 2\Delta\Theta} - 2e^{-F_n^2\Delta\Theta} + 1)]\right\} \tag{A1}$$

where

$$K = \frac{\Delta\Theta}{2}\left(\frac{1}{N} + 1\right) + \frac{1}{3} - \frac{1}{45\,\Delta\Theta} + \frac{2}{\Delta\Theta}\sum_{n=1}^{\infty}\frac{e^{-F_n^2\Delta\Theta}}{F_n^4}.$$

For $m \leqslant r$,

$$Q_{m,r} = \frac{1}{K}\left[\frac{\Delta\Theta}{2N}\left(Q_{g_{m,0}} + 2\sum_{s=1}^{r-1} Q_{g_{m,s}} + Q_{g_{m,r}}\right) - \frac{\Delta\Theta}{N}\sum_{t=1}^{r-1} Q_{m,t} - \frac{\Delta\Theta}{2}\left(Q_{g_{0,r-m}} + 2Q_{1,r-m+1} + 2Q_{2,r-m+2} + \dots + 2Q_{m-2,r-2} + 2Q_{m-1,r-1}\right) - 2\sum_{n=1}^{\infty}\frac{1}{F_n^4\Delta\Theta}\right.$$
$$\left.\{Q_{g_{0,r-m}}[e^{-F_n^2 m\Delta\Theta}(-F_n^2\,\Delta\Theta - 1) + e^{-F_n^2(m-1)\Delta\Theta}] + Q_{1,r-m+1}[e^{-F_n^2 m\Delta\Theta} - 2e^{-F_n^2(m-1)\Delta\Theta} + e^{-F_n^2(m-2)\Delta\Theta}] + Q_{2,r-m+2}[e^{-F_n^2(m-1)\Delta\Theta} - 2e^{-F_n^2(m-2)\Delta\Theta} + e^{-F_n^2(m-3)\Delta\Theta}] + \dots + Q_{m-1,r-1}(e^{-F_n^2 2\Delta\Theta} - 2e^{-F_n^2\Delta\Theta} + 1)\}\right] \tag{A2}$$

Résumé—La convection forcée entre deux plaques parallèles est étudiée dans le cas d'un fluide à propriétés constantes. Les plaques ont une capacité thermique finie et la chaleur leur est fournie suivant l'axe, simultanément, de façon arbitraire. La paroi est supposée suffisamment mince ou très conductrice de façon à pouvoir négliger la variation de température dans l'épaisseur. La température du fluide est variable dans la section de la conduite, mais on suppose la vitesse du fluide constante. L'équation de l'énergie est laminaire. On donne une méthode générale de résolution et quelques exemples. En particulier, on traite le cas d'un chauffage uniforme de la paroi avec variation sinusoïdale en fonction du temps et celui d'un chauffage sinusoïdal en fonction de la distance axiale et exponential en fonction du temps.

Zusammenfassung—Der Wärmeübergang an eine Flüssigkeit mit gleichbleibenden Stoffwerten wird für Zwangskonvektion zwischen zwei parallelen Platten analysiert. Die Platten besitzen eine endliche Wärmekapazität und es wird ihnen sowohl hinsichtlich der Zeit als auch der achsialen Richtung beliebig Wärme zugeführt. Die Wand lässt sich als genügend dünn oder gut leitend annehmen, sodass die Temperaturänderung innerhalb der Wanddicke vernachlässigt werden kann. Die Flüssigkeitstemperatur ist veränderlich über den Kanalquerschnitt, doch ist die Strömungsgeschwindigkeit im ganzen Kanal als konstant angenommen (Kolbenströmungsbedingung). Die Energiegleichung ist laminar. Eine allgemeine Lösungsmethode ist angegeben und einige anschauliche Beispiele sind ausgeführt. Sie umfassen gleichmässige Wandheizung mit sinusförmiger Zeitabhängigkeit bzw. längs des Kanals sinusförmig veränderliche Beheizung und exponentielle Zeitabhängigkeit.

Аннотация—Рассматривается перенос тепла в среду с постоянными свойствами при вынужденной конвекции между двумя параллельными пластинами, имеющими конечную теплоемкость. Тепло подводится к пластинам произвольно как по оси, так и во времени. Стенка предполагается достаточно тонкой (хорошо проводит тепло), так что температурными изменениями по ее толщине можно пренебречь. Температура среды переменна в поперечном сечении, а скорость ее предполагается постоянной во всем канале (условие ползущего течения). Уравнение энергии является ламинарным. Дается общий метод решения, иллюстрируемый несколькими примерами. В примерах рассматривается равномерный нагрев стенки, изменяющийся синусоидально во времени, и нагрев, изменяющийся синусоидально по оси и экспоненциально во времени.